Readers' Forum

Brief discussions of previous investigations in the aerospace sciences and technical comments on papers published in the AIAA Journal *are presented in this special department. Entries must be restricted to a maximum of 1000 words, or the equivalent of one* Journal *page including formulas and figures. A discussion will be published as quickly as possible after receipt of the manuscript. Neither the AIAA nor its editors are responsible for the opinions expressed by the correspondents. Authors will be invited to reply promptly.*

Comment on "Centerline Formulation in the Numerical Computation of Axisymmetric Flows" and "Conservation Errors in Axisymmetric Finite Difference Equations"

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REFERENCES 1 and 2 have addressed the problem of the finite difference formulation of the centerline boundary in axisymmetric flows. The finite difference form of the governing equations can be obtained either through a Taylor series expansion or by a control volume analysis, but in either case careful attention must be paid to the order of accuracy of the finite difference approximation. We will show that the method of Ref. 1 introduces numerical diffusion resulting in a lower order of accuracy compared with the second-order finite difference method. Furthermore, we show that the results of Ref. 2 are wrong because conservation of mass is violated.

If a second-order central differencing method is used, the removable singularity in the governing equation along the centerline must first be treated. This can be done in a straightforward manner by deriving the limiting form of the governing equation on the axis, with the result given by Eq. (4) of Ref. 1. The equation that results is exact in the appropriate limit. If a second-order-accurate central differencing scheme is used, the resulting finite difference equation is, in the notation used by Sukanek and Rhodes[1]

$$F_{n+1,0}=F_{n,0}+\frac{4\mu_{n,0}\Delta x}{b(\Delta\psi)^2}(F_{n,1}-F_{n,0})+\Delta x\, P_{n,0} \tag{1}$$

Instead of this approach, Sukanek and Rhodes[1] choose a control volume method, evaluating the dependent variables in their Eq. (4) at the centerline and taking the region of integration to be that from the centerline to the point halfway between the center and the first grid point. Using this method, they obtain the equation

$$F_{n+1,0}=F_{n,0}+\frac{8a_+\Delta x}{(\Delta\psi)^3 b_+}(F_{n,1}-F_{n,0})+\Delta x P_{n,0} \tag{2}$$

corresponding to Eq. (1). Equation (2) here is Eq. (7) of Ref. 1 with several typographical errors corrected.

This equation does not retain second-order accuracy, as is easily shown by applying a Taylor series expansion to F in Eqs. (1) and (2) and using the definitions of a_+ and b_+ as given in Ref. 1. The result, in the case of Eq. (1), is Eq. (4) of Ref. (1), while for Eq. (2), the result is

$$\frac{\partial F}{\partial x}=\frac{2\mu_0}{b}\frac{\partial^2 F}{\partial\psi^2}\Big)_{\psi=0}+\frac{\mu_1-\mu_0}{b}\frac{\partial^2 F}{\partial\psi^2}\Big)_{\psi=0}+P\Big)_{\psi=0} \tag{3}$$

Comparing this result to Eq. (4) of Ref. 1 shows that the reduced order of accuracy of the control volume approach used by Sukanek and Rhodes[1] leads to a numerical diffusion equal to

$$(\mu_1-\mu_0)\,\frac{\partial^2 F}{\partial\psi^2}\Big)_{\psi=0} \tag{4}$$

In a companion note, Sukanek[2] argues that the control volume analysis used in Ref. 1 (and thus its inherent numerical diffusion) is required to assure global conservation. This argument rests on an incorrect statement of global conservation. Sukanek writes the finite difference analog for the expression of global conservation of a variable F as

$$\sum_{n=0}^{N}\left\{\frac{1}{\psi}\frac{\partial}{\partial\psi}\left(a\frac{\partial F}{\partial\psi}\right)\right\}\Psi_n\Delta\psi=0 \tag{5}$$

Global conservation is established through the integrated mass flux of the appropriate conservation variable. Thus, if we take F, for example, to be the velocity, then we seek the integrated flux of momentum $\dot M$

$$\dot M=2\pi\int_0^{r_e}\rho u u r\,\mathrm{d}r=2\pi\int_0^{\psi_e}u\psi\,\mathrm{d}\psi \tag{6}$$

using the stream function transformation of Ref. 1 and 2. With no loss of generality, we can take the pressure to be constant, and for inviscid flow along $\psi=\psi_e$ we can obtain from Eq. (1) of Ref. 2

$$\frac{1}{2\pi}\frac{\mathrm{d}\dot M}{\mathrm{d}x}=\int_0^{\psi_e}\frac{\partial U}{\partial x}\psi\,\mathrm{d}\psi=\int_0^{\psi_e}\frac{\partial}{\partial\psi}\left(a\frac{\partial u}{\partial\psi}\right)\mathrm{d}\psi \tag{7}$$

The right-hand side of Eq. (7) can be integrated directly, and since $a(\partial u/\partial\psi)$ is zero at $\psi=0$ and by definition at $\psi=\psi_e$, the constant momentum flux condition ($\mathrm{d}\dot M/\mathrm{d}x=0$) appropriate to the problem is obtained. Clearly the appropriate conservation expression is thus

$$\int_0^{\psi_e}\frac{\partial}{\partial\psi}\left(a\frac{\partial F}{\partial\psi}\right)\mathrm{d}\psi=0$$

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for a general conservation variable F, and thus the appropriate finite difference analog is

$$\sum_{n=0}^{N} \frac{\partial}{\partial \psi}\left(a\frac{\partial F}{\partial \psi}\right)\Delta\psi = 0 \qquad (8)$$

and not Eq. (5) as used by Sukanek.[2]

References

[1] Sukanek, P.C., and Rhodes, R.P., "Centerline Formulation in the Numerical Computation of Axisymmetric Flows," *AIAA Journal,* Vol. 16, Oct. 1978, pp. 1099-1101.

[2] Sukanek, P.C., "Conservation Errors in Axisymmetric Finite-Difference Equations," *AIAA Journal,* Vol. 17, Jan. 1979, pp. 99-101.

Reply by Author to R. Edelman and P.T. Harsha

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EDELMAN and Harsha present two objections to our work. They claim, first, that the centerline formulation reported in Ref. 1 is not second-order accurate and introduces a "spurious" diffusion term; second, that the conservation statement employed in Ref. 2 is incorrect. These objections are interrelated, since it is our contention that the "spurious" diffusion term is necessary to preserve conservation.

The finite differrence analog of the global conservation property used in Ref. 2 is identical to that reported by Edelman and Harsha, provided a consistent choice of ψ_n is employed:

$$\sum_{n=0}^{N}\left\{\frac{1}{\psi}\frac{\partial}{\partial\psi}\left(a\frac{\partial F}{\partial\psi}\right)\right\}_n\psi_n\Delta\psi \equiv \sum_{n=0}^{N}\left\{\frac{\partial}{\partial\psi}\left(a\frac{\partial F}{\partial\psi}\right)\right\}_n\Delta\psi \qquad (1)$$

The notation $\{\ \}_n$ refers to the finite difference form of the quantity within the braces. The left side of this identity was used in Ref. 2 to facilitate the examination of conservation when a limiting form of the equation is used at the axis of symmetry. In this case, the quantity on the left side in braces must be replaced by its limit as ψ goes to zero.

The right or left side of Eq. (1) may be used to obtain an expression for the centerline which guarantees overall conservation. Both give the same result, Eq. (13) of Ref. 2, indicating that the appropriate value of the parameter a, the diffusion coefficient, is the average of a between the centerline and the first radial position. This is in agreement with the centerline formulation derived in Ref. 1 using an integral approach. The same formulation can be found using the control volume method, i.e., expressing the balance among convection, diffusion, and production of the quantity F for the center streamtube. The diffusion term must be evaluated at the streamtube boundary. Finally, the result also agrees with the intuitive concept that the flux of material leaving tube n must equal the flux entering tube $(n+1)$. A limiting form of the conservation equation, where the diffusion coefficient is evaluated at the centerline and not the streamtube boundary, does not permit this equality.

Edelman and Harsha use the limiting form of the governing equations at the centerline. This form is exact only for each point on the axis of symmetry. However, the finite difference form of the equation must be valid not for a point, but for some finite region surrounding the point. In the appropriate limits, Eq. (7) of Ref. 1 does reduce to the correct equation:

$$\lim_{\substack{\Delta X \to 0\\ \Delta\psi\to 0}}\left[\frac{F_{n+1,0}-F_{n,0}}{\Delta X} - \frac{4a_+}{b\Delta\psi}\frac{2(F_{n,1}-F_{n,0})}{(\Delta\psi)^2} - P_{n,0}\right]$$

$$= \frac{\partial F}{\partial X} - \frac{2\mu_o}{b}\frac{\partial^2 F}{\partial\psi^2} - P \qquad (2)$$

While the limiting form of the conservation equations at the centerline is exact, this alone does not guarantee that the corresponding finite difference equations will also be exact.

References

[1] Sukanek, P.C. and Rhodes, R.P., "Centerline Formulation in the Numerical Computation of Axisymmetric Flows," *AIAA Journal,* Vol. 16, Oct. 1978, pp. 1099-1101.

[2] Sukanek, P.C., "Conservation Errors in Axisymmetric Finite-Difference Equations," *AIAA Journal,* Vol. 17, Jan. 1979, pp. 99-101.

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